Point processes in random environment and application to the study of longevity risk

Sarah Kaakaï, Le Mans Université

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Some application of point processes

- Renewal of interest in **point processes** in the past years.
- Flexibility allows for the modeling of a wide range of phenomena in:
  - Finance and Insurance (Dassios and Zho (2011), Giesecke and Kim (2011), Bacry et al. (2015), Jaisson and Rosembaum (2015), El Euch et. al. (2016)...)
  - Neurosciences (Reynaut Bouret et al. (2013), Chevallier et al. (2015), Galves and Löcherbach (2016)...)
  - Individual-based model in biology and ecology (Fournier and Méléard (2004), Champagnat et al. (2006), Méléard and Tran (2010), Billard et al (2016)...)
  - Chemical reactions (Andersen and Kurtz (2015))
  - Epidemiology, cyber risk...
- Human longevity?

Point processes and longevity

- **Observed data**: population data (birth, death count).
  ↳ longevity indicators (life expectancy, death rates,...) are a by-product of the population dynamics.
- Individual-based model for human populations (Bensusan (2010), Boumezoued and El Karoui (2016)).
- Issues:
  - Non-stationarity, influence of macro-environment (El Karoui et al. (2018)).
  - High heterogeneity: Structured population.
  - Interactions $\Rightarrow$ non-linearity.
- Need for probabilistic tools to deal with this complexity.

Point processes

- General theory on point processes measures : 60-70's.
- Several viewpoint to define point processes:
  - Random counting measures (static).
  - Random sets (static).
  - In certain cases: Counting processes (multivariate, marked),

    Dynamic viewpoint.
- Central concept: **Intensity** (measure).

  But not sufficient to defined point processes in general settings.
- Pathwise construction.

1 Some generalities on point processes

2 Pathwise representation of point processes in random environment
- Process with bounded intensity
- General case
- A first “converse result”

3 Strong comparison of point processes
- Point processes with ordered intensities
- General case
- Converse result

4 Birth-Death-Swap process in random environment

Outline

1 Some generalities on point processes

2 Pathwise representation of point processes in random environment

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Notations

- Filtered probability space $(\Omega, (\mathcal{G}_t), \mathrm{P})$.
- E Polish space and $\mathcal{E}$ its Borel σ-algebra.
- $\mathcal{P}(\mathcal{G}_t)$ σ-algebra of predictable processes generated by processes

$$C_t = H\mathbb{1}_{]t_0,t_1]}(t), \quad t \geqslant 0,\ H \in \mathcal{G}_{t_0}.$$

Random counting measure

Definition (Random counting measure/Point process (simple))

A random counting measure is a random measure $M : \Omega \times \mathcal{E} \to \bar{\mathbb{N}}$ such that $M_\omega : A \in \mathcal{E} \mapsto M(\omega, A)$ is a purely atomic and its every atom has weight one, a.s.

- Equivalent viewpoint: M is a r.v taking values in the space of counting measures.
- **Mean measure** of M:

$$\mu(A) = \mathrm{E}[M(A)].$$

- For any nonnegative measurable function f ($f \in \mathcal{E}_+$),

$$M(f)(\omega) = \int_E f(x) M(\omega, \mathrm{d}x).$$

Some properties of random measures

- EXAMPLE M is a Poisson random measure with measure μ if:
  - $\forall A \in \mathcal{E}$, $M(A) \sim \mathcal{P}(\mu(A))$
  - if $A_1, \ldots, A_p$ are disjoints then $M(A_1), \ldots, M(A_p)$ are independent.

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- PROPOSITION 1 The probability law of a random (counting) measure M on $(E, \mathcal{E})$ is *uniquely* determined by its Laplace functional,

$$\mathrm{E}[\exp(-M(f))], \quad f \in \mathcal{E}_+.$$

Poisson measures: $\mathrm{E}[\exp(-M(f))] = \exp(-\int(1 - e^f)\mathrm{d}\mu)$.

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- PROPOSITION 2 The probability law of a random counting measure M on $(E, \mathcal{E})$ is *uniquely* determined by the set of avoidance probabilities:

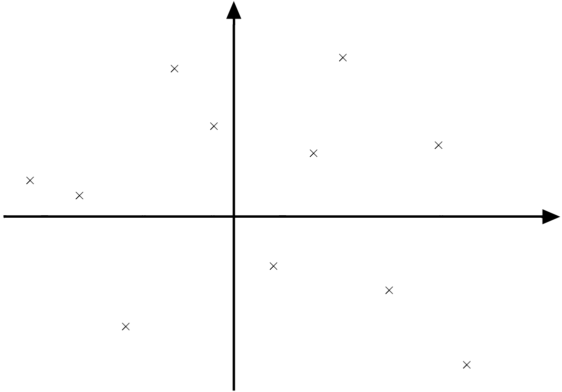
$$\mathrm{P}(M(A) = 0), \quad A \in \mathcal{E}.$$

If $P(M(A) = 0) = e^{-\mu(A)}$ with μ sigma-finite, then M is a Poisson measure.

Point process viewpoint

- The point process associated with a sequence (X_i) of $(E, \mathcal{E})$-random variables is defined for all $A \in \mathcal{E}$ by

$$M(A) = \sum_i \mathbb{1}_A(X_i), \quad (M = \sum_i \delta_{X_i}).$$



- Random set viewpoint: $M(\omega) = \{x \in E;\ x = X_i(\omega) \text{ for some i}\}$.

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- $M(f) = \int_E f(x) M(\omega, \mathrm{d}x) = \sum_i f(X_i)$.
- PROPOSITION: If M is a random counting measure and μ is σ-finite,

  $$M = \sum_i \delta_{X_i},$$

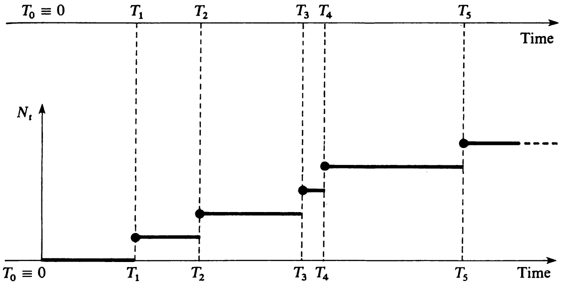
  with (X_i) a sequence of r.v taking values on a enlargement $(\bar{E}, \bar{\mathcal{E}})$ of $(E, \mathcal{E})$.

Point processes on the half line

Case $E = \mathbb{R}^+$.

- Atoms can be ordered: $(X_i) \to$ increasing sequence $(T_n)_{n\in\mathbb{N}}$.
- Associated **counting process**

$$N_t = N(]0,t]) = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}}.$$



(Source: Bremaud)

Point processes on the half line

Case $E = \mathbb{R}^+$: dynamic viewpoint.

- Associated **counting process**

$$N_t = N(]0,t]) = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}}.$$

- The random measure is seen as a r.v taking values in the space $\mathcal{A}$ of counting functions.
- If N is adapted to a filtration $(\mathcal{G}_t)$, for $C \in \mathcal{P}(\mathcal{G}_t)$

$$\int_0^t C_s \mathrm{d}N_s = \sum_{n \geqslant 0} C_{T_n} \mathbb{1}_{\{T_n \leqslant t\}}$$

Poisson process

- DYNAMIC VIEWPOINT Let N be a counting process adapted to a filtration $(\mathcal{G}_t)$. Then N is a $(\mathcal{G}_t)$-**Poisson process** if:
  (i) Independent increments:
  $$N_{s+h} - N_s \perp\!\!\!\perp \mathcal{G}_s.$$
  (ii) Stationary increments:
  $$N(]s, s+h]) = N_{s+h} - N_s \overset{d}{=} N_h (\sim \mathcal{P}(\lambda h)).$$
- POINT PROCESS VIEWPOINT
  $$N = \sum_{n=1}^{\infty} \mathbb{1}_{\{T_n \leqslant t\}}, \quad (S_n)_{n \geqslant 1} = (T_n - T_{n-1})_{n \geqslant 1} \text{ i.i.d } \sim \mathcal{E}(\lambda).$$
- RANDOM MEASURE VIEWPOINT For any $f \in \mathcal{E}_+$,
  $$\mathrm{E}[e^{-N(f)}] = \exp(-\int_{\mathbb{R}^+} (1 - e^{-f(s)})\nu(\mathrm{d}s)), \quad \nu = \lambda \text{ Leb.}$$

Intensity and martingale property

Let N be a $(\mathcal{G}_t)$ Poisson process with intensity λ.

- Let $C_s = \mathbf{1}_H \mathbf{1}_{]t_0,t_1]}(s)$ with $H \in \mathcal{G}_{t_0}$. Then,

$$\mathsf{E}[\int_0^\infty C_s \mathrm{d}N_s] = \mathsf{E}[\mathbf{1}_H (N_{t_1} - N_{t_0})] = \mathsf{E}[\mathbf{1}_H]\lambda(t_1 - t_0) = \mathsf{E}[\int_0^\infty C_s \lambda \mathrm{d}s]$$

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- Generalization to nonnegative **predictable processes** $C \in \mathcal{P}(\mathcal{G}_t)$:

$$\mathsf{E}[\int_0^\infty C_s \mathrm{d}N_s] = \mathsf{E}[\int_0^\infty C_s \lambda \mathrm{d}s]. \tag{1}$$

- Equivalent property:

$(N_t - \lambda t)$ is a $(\mathcal{G}_t)$-martingale.

- (1) can be generalized to define the intensity process of a counting process.

General case: stochastic intensity

Definition (Stochastic intensity (Bremaud))

Let N be an adapted counting process, and (λ_t) a nonnegative $(\mathcal{G}_t)$-process with

$$\forall t \geqslant 0, \quad \int_0^t \lambda_s \mathrm{d}s < \infty a.s.. \tag{2}$$

Then, N admits the $(\mathcal{G}_t)$-**intensity** (λ_t) if for all nonnegative predictable process C

$$\mathsf{E}[\int_0^\infty C_s \mathrm{d}N_s] = \mathsf{E}[\int_0^\infty C_s \lambda_s \mathrm{d}s]. \tag{3}$$

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- (2) is a non-explosion condition.
- Equivalenty: $(N_t - \int_0^t \lambda_s \mathrm{d}s)$ is a $(\mathcal{G}_t)$ local martingale.
- The existence of an intensity has to do with the absolute continuity of the predictable compensator A of N w.r.t Leb. If λ exists, $A_t = \int_0^t \lambda_s \mathrm{d}s$.

Examples and uniqueness

- Non-Homogeneous Poisson processes (NHP) $(\lambda_t) = (f(t))$ is a deterministic function.
- Cox/Doubly stochastic Poisson processes

$$(\lambda_t) \text{ is } \mathcal{G}_0\text{-measurable.}$$

$(N_t - N_s$ is independent of $\mathcal{G}_s$ *given* $\mathcal{G}_0)$

- **Uniqueness of intensity process** In order to be unique, (λ_t) should be taken predictable.

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- **Uniqueness of intensity process** In order to be unique, (λ_t) should be taken predictable.
- Does the intensity process characterizes the probability distribution of a counting process?

Martingale characterization

Let us come back to the example of the Poisson process:

Theorem (Martingale characterization of Poisson processes (Watanabe))

Let N be a counting process and f a locally integrable nonnegative function such that $N - \int_0^t f(s)\mathrm{d}s$ is $(\mathcal{G}_t)$-martingale.
Then N is a $(\mathcal{G}_t)$-non homogeneous Poisson process of intensity function f.

- **Extension** The result still holds for Cox processes (Bremaud)
- Sketch of the proof:
  - Use (3) to show that $(Z_t) = (e^{iuN_t - \int_0^t (e^{iu}-1)f(s)\mathrm{d}s})$ is a $(\mathcal{G}_t)$-martingale.
  - Laplace functional characterization.

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- Intensity process is often a function of the counting process itself.
- EXAMPLES
  - PURE BIRTH PROCESS WITH IMMIGRATION: $\lambda_s = a + bN_s$.

    Such a process is a Continuous Time Markov Chain (CTMC).

    $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are independent and $T_{k+1} - T_k \sim \mathcal{E}(a + kb)$.
  - (LINEAR) HAWKES PROCESS:

$$\lambda_s = a + \int_0^s h(s-r)\mathrm{d}N_r.$$

$\lambda_s = f([N]_{s-})$, with $[N]_s = (N_{t\wedge s})_{t\geqslant 0}$ and $f : n \in \mathcal{A} \mapsto (a + \int_0^s h(s-r)dn(r))$.

In particular, if $(f([N]_{t-}))$ is the $(\mathcal{G}_t)$-intensity of N then $(f([N]_{t-}))$ is also its $(\mathcal{F}_t^N)$ intensity.

General case(II)

Theorem (Jacod 75 (partial))

Let $\lambda : \Omega \times \mathbb{R}^+ \times \mathcal{A} \to \mathbb{R}^+$ a $(\mathcal{G}_t)$-predictable functional such that for all counting path $n \in \mathcal{A}$ and $t \geqslant 0$, $\lambda(\omega, t, [n]) \in \mathcal{G}_0$.
Then if two counting processes N and N' have respective $(\mathcal{G}_t)$ intensities $(\lambda(\omega, t, [N]_{t-}))$ and $(\lambda(\omega, t, [N']_{t-}))$, then N and N' have the same distribution.

- Also weak existence result (under additional assumptions).

General case(II)

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- Also weak existence result (under additional assumptions).

BUT

Random environment/external noise is $\mathcal{G}_0$-measurable "known at time 0".

↳ Stochastic intensity not sufficient to introduce a counting process.

Space-time point process

Result originally stated for Marked point processes.

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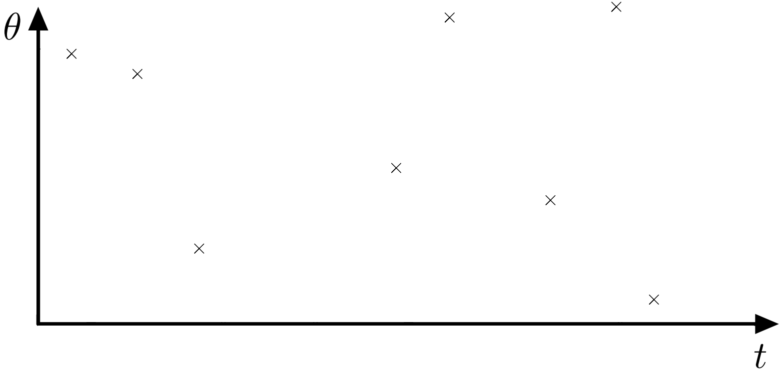
- SPACE-TIME POINT PROCESS
  - Random counting measure N on $\mathbb{R}^+ \times E$.
  - Defined relatively to $(\mathcal{G}_t)$ if for all $A \in \mathcal{E}$, $N([0,t] \times A) \in \mathcal{G}_t$



Marked point process

- Space-time point process
  - Random counting measure N on $\mathbb{R}^+ \times E$.
  - Defined relatively to $(\mathcal{G}_t)$ if for all $A \in \mathcal{E}$, $N([0,t] \times A) \in \mathcal{G}_t$
- Marked point process on $(\mathbb{R}^+ \times E, \mathcal{B}_{\mathbb{R}^+} \times \mathcal{E})$.
  - $$N_t(A) = N([0,t] \times A) = \sum_n \mathbb{1}_{\{T_n \leqslant t\}} \mathbb{1}_A(X_n),$$

    with (T_n) increasing sequence and $(X_n)_{n \geqslant 0}$ sequence of r.v taking values in the **mark space** E.
  - In particular, $(N_t(E))$ is a counting process.
- All space-time point processes are not Marked point processes!

Examples (Marked Point processes)

- MULTIVARIATE COUNTING PROCESS $E = \{x_1, \dots, x_p\}$

$$N_t^i = N_t(\{i\}) = \sum_n \mathbb{1}_{\{T_n \leqslant t\}} \mathbb{1}_{\{X_n = x_i\}}, \quad i = 1, \dots, p.$$

  - Pure birth marked by discrete frailty variables, interacting Hawkes processes...
  - Usual hypothesis: components of $(N^1, \dots, N^p)$ have no jumps in common.

- COMPOUND POISSON PROCESS (T_n) are jump times of a Poisson process, marks (X_i) are i.i.d $\sim \pi$ and $\perp\!\!\!\perp$ of (T_n).
  A compound Poisson process is a Poisson random measure of mean measure $\lambda \mathrm{d}s \otimes \pi$.

$(\mathcal{G}_t)$ Poisson measures (I)

Let γ be a sigma-finite measure on $(E, \mathcal{E})$.

Definition (Space-time $(\mathcal{G}_t)$ Poisson measure)

$Q(\mathrm{d}t, \mathrm{d}x)$ is a $(\mathcal{G}_t)$-Poisson measure on $\mathbb{R}^+ \times E$ of mean measure $\mathrm{d}t \otimes \gamma(\mathrm{d}x)$ iff $\forall A_1, ..., A_n$ disjoint sets with $\gamma(A_i) < \infty$, the counting processes $Q_t(A_i)$ defined for $i = 1..p$ by

$$Q_t(A_i) = Q([0, t] \times A_i), \quad \forall t \geqslant 0$$

are independent $\mathcal{G}_t$-Poisson processes of intensity $\gamma(A_i)$.

- If $E = \{x\}$, Q is a $(\mathcal{G}_t)$-Poisson process of intensity $\gamma(\{x\})$.
- When γ is finite, $Q = \{(T_n, X_n)\}$ is a compound Poisson process. It particular, its jump times can be enumerated increasingly.

It is not the case when γ is only sigma-finite.

$(\mathcal{G}_t)$ Poisson measures (II)

Definition (Space-time $(\mathcal{G}_t)$ Poisson measure)

$Q(\mathrm{d}t, \mathrm{d}x)$ is a $(\mathcal{G}_t)$-Poisson measure on $\mathbb{R}^+ \times E$ of mean measure $\mathrm{Leb}(\mathrm{d}t)\gamma(\mathrm{d}x)$ iff $\forall A_1, ..., A_n$ disjoint sets with $\gamma(A_i) < \infty$, the counting processes $Q_t(A_i)$ defined for $i = 1..p$ by

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$$\mathsf{E}[Q_t(A)] = t\gamma(A) \quad \Leftrightarrow$$

$$\mathsf{E}[\int_{\mathbb{R}^+ \times E} \mathbb{1}_{[0,t]\times A}(s,x) Q(\mathrm{d}s, \mathrm{d}x)] = \mathsf{E}[\int_{\mathbb{R}^+ \times E} \mathbb{1}_{[0,t]\times A}(s,x)\gamma(\mathrm{d}x)\mathrm{d}s].$$

- **Generalization** to predictable processes $(G(s,x)) \in \mathcal{P}(\mathcal{G}_t) \otimes \mathcal{E}$

$$\mathsf{E}[\int_{\mathbb{R}^+ \times E} G(s,x) Q(\mathrm{d}s, \mathrm{d}x)] = \mathsf{E}[\int_{\mathbb{R}^+ \times E} G(s,x)\gamma(\mathrm{d}x)\mathrm{d}s].$$

Stochastic intensity

STOCHASTIC INTENSITY OF SPACE-TIME POINT PROCESS N

Predictable random measure $\lambda(\omega, s, \mathrm{d}x)$ such that for all $G \in \mathcal{P}(\mathcal{G}_t) \otimes \mathcal{E}$

$$\mathrm{E}[\int_{\mathbb{R}^+ \times E} G(s,z) N(\mathrm{d}s, \mathrm{d}z)] = \mathrm{E}[\int_{\mathbb{R}^+ \times E} G(s,z) \lambda(s, \mathrm{d}x) \mathrm{d}s]. \tag{4}$$

Equivalently,

The counting process $(N_t(A))$ has the $(\mathcal{G}_t)$ intensity $(\lambda_t(A)) = (\int_A \lambda(t, \mathrm{d}x))$.

- $(\mathcal{G}_t)$ intensity of Poisson random measure : $\lambda(s, \mathrm{d}x) = \gamma(\mathrm{d}x)$.
- The $(\mathcal{G}_t)$ multivariate intensity of a multivariate counting process $\mathbf{N} = (N^1, \ldots, N^p)$ is the vector $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^p)$ of the components $(\mathcal{G}_t)$-intensity processes.

Characterization of space-time point processes

- A Poisson measure is uniquely characterized by its intensity.
- If $\mathcal{G}_t = \mathcal{G}_0 \vee \sigma(N_s(B) : s \leqslant t, \ B \in \mathcal{E})$, then two marked point processes with $(\mathcal{G}_t)$ intensity $(\lambda(t, \mathrm{d}x))$ have the same distribution.

But

- Poor structure random environment/external noise which is $\mathcal{G}_0$-measurable.
- Existence results in distribution, no pathwise construction.

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Pathwise representation of point processes

Two main approaches:

- Multiple random time changes (Kurtz (1980), Garcia (1995), Garcia and Kurtz (2008)).
- Thinning and projection of Poisson measures
  - Population dynamics (Fournier and Méléard (2004), Garcia and Kurtz (2006), Méléard and Tran (2009), El Karoui and Boumezoued (2016)), interacting Hawkes processes (Chevallier et. al (2015), Delattre et al (2016),..), PDP (Lemaire et al. (2018)...)
  - General construction in the spirit of Massoulié (1998): Point processes described by an intensity process+ Poisson measure.

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First step: *How to simulate a counting process with bounded intensity $\lambda_t \leqslant \bar{\lambda}$?*

- Naive idea: take $N_t^{\lambda} = \int_0^t \frac{\lambda_s}{\bar{\lambda}} dN_s$ with N Poisson process of intensity $\bar{\lambda}$. $(N_t^{\lambda} - \int_0^t \lambda_s ds)_{t \geqslant 0}$ is a local martingale.

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  But N^{λ} is **not a counting process**.

- Solution: Increase space dimension and represent space-time point processes as strong solutions of SDEs driven by Poisson measures.

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Thinning with bounded intensity

Simulation of counting process N with given (stochastic) bounded intensity $\lambda_t \leqslant \bar{\lambda}$ (Lewis and Shedler (1979)):

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Simulation of counting process N with given (stochastic) bounded intensity $\lambda_t \leqslant \bar{\lambda}$ (Lewis and Shedler (1979)):

1. Take Q Marked Poisson measure on $\mathbb{R}^+ \times [0,1]$ with intensity $\bar{\lambda} ds \otimes du$:

$$Q(ds, du) = \sum_{n \geqslant 1} \delta_{T_n}(s) \delta_{U_n}(u)$$

   with

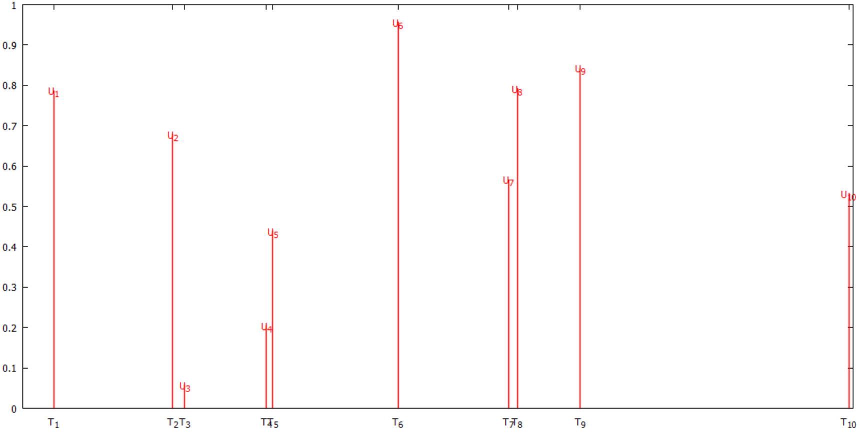
   - $(T_n)_{n \geqslant 1}$ jump times of Poisson process of intensity $\bar{\lambda}$.
   - $(U_n)_{n \geqslant 1}$ i.i.d variables on $[0,1]$.

2. Accept T_n as jump time of N if

$$U_n \leqslant \frac{\lambda_{T_n}}{\bar{\lambda}}.$$

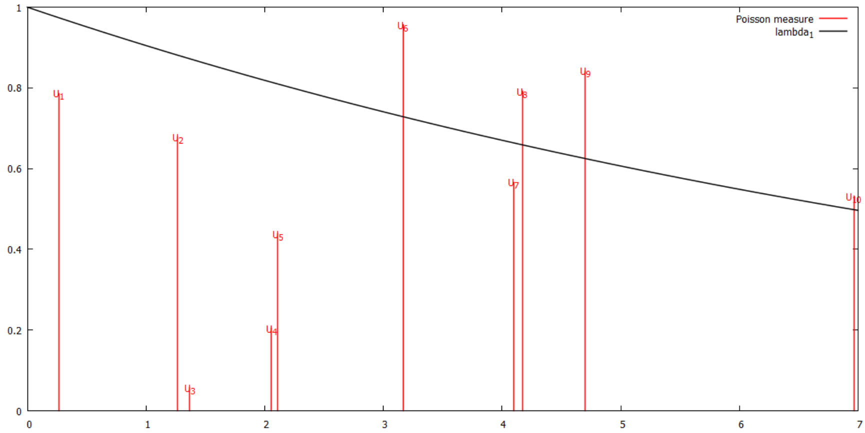
NHP with intensity $\lambda_t = e^{-\beta t}$

Step 1: Simulation of $\{(T_n, U_n)_{n \geqslant 0}\}$.



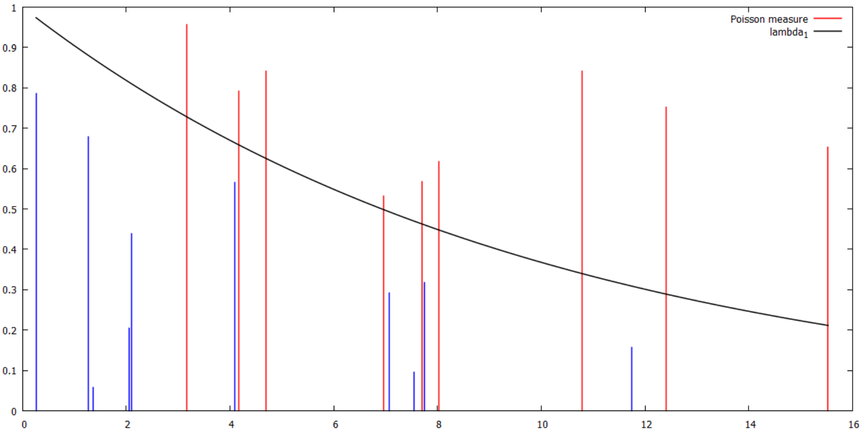
NHP with intensity $\lambda_t = e^{-\beta t}$

Step 2: $U_n \leqslant \frac{\lambda_{T_n}}{\bar{\lambda}}$.



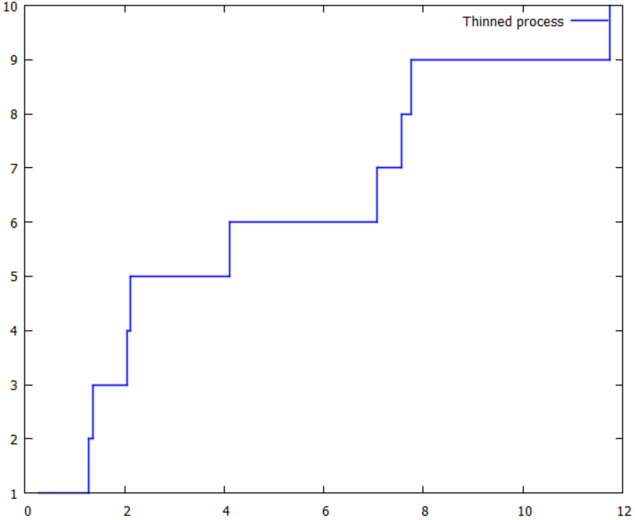
NHP with intensity $\lambda_t = e^{-\beta t}$

Step 2: $U_n \leqslant \frac{\lambda_{T_n}}{\bar{\lambda}}$.



NHP with intensity $\lambda_t = e^{-\beta t}$

Step 3: Projection on $\mathbb{R}^+$



Thinning equation: bounded case (I)

$$N_t = \sum_{n \geqslant 1} \mathbb{1}_{\{T_n \leqslant t\}} \mathbb{1}_{\{U_n \leqslant \frac{\lambda_{T_n}}{\bar{\lambda}}\}} = \sum_{n \geqslant 1} \mathbb{1}_{\{T_n \leqslant t\}} \mathbb{1}_{D^\lambda}(T_n, U_n),$$

with D^λ the predictable set:

$$D^\lambda = \{(s, u);\ u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}.$$

Thinning equation: bounded case (I)

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with D^λ the predictable set:

$$D^\lambda = \{(s,u);\ u \leqslant \frac{\lambda_s}{\bar\lambda}\}.$$

Thinning equation associated with simulation

Step 2 Projection of Q on D^λ:

$$Q^D(ds,du) = \mathbb{1}_{D^\lambda}(s,u) Q(ds,du) = \sum_{n\geqslant 1} \mathbb{1}_{D^\lambda}(T_n,U_n)\delta_{T_n}(s)\delta_{U_n}(u).$$

Step 3 Projection on $\mathbb{R}^+$:

$$N_t^\lambda = \int_0^t \int_{[0,1]} Q^D(ds,du) = \int_0^t \int_{[0,1]} \mathbb{1}_{\{u\leqslant \frac{\lambda_s}{\bar\lambda}\}} Q(ds,du).$$

Thinning equation: bounded case (II)

- Stochastic intensity: Let $C \in \mathcal{P}(\mathcal{G}_t)$

$$\mathsf{E}\Big[\int_0^t C_s \mathrm{d} N_s^{\lambda}\Big] = \mathsf{E}\Big[\int_0^t \int_{[0,1]} C_s \mathbb{1}_{\{u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}} Q(ds, du)\Big]$$

- Stochastic intensity: Let $C \in \mathcal{P}(\mathcal{G}_t)$

$$\mathsf{E}[\int_0^t C_s \mathrm{d}N_s^{\lambda}] = \mathsf{E}[\int_0^t \int_{[0,1]} C_s \mathbb{1}_{\{u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}} Q(ds, du)]$$

$$= \mathsf{E}[\int_0^t \int_{[0,1]} C_s \mathbb{1}_{\{u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}} \bar{\lambda} \mathrm{d}u \mathrm{d}s]$$

- Stochastic intensity: Let $C \in \mathcal{P}(\mathcal{G}_t)$

$$
\begin{aligned}
\mathsf{E}[\int_0^t C_s \mathrm{d}N_s^{\lambda}] &= \mathsf{E}[\int_0^t \int_{[0,1]} C_s \mathbb{1}_{\{u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}} Q(ds, du)] \\
&= \mathsf{E}[\int_0^t \int_{[0,1]} C_s \mathbb{1}_{\{u \leqslant \frac{\lambda_s}{\bar{\lambda}}\}} \bar{\lambda} \mathrm{d}u \mathrm{d}s] \\
&= \mathsf{E}[\int_0^t \mathrm{d}s \left(C_s \bar{\lambda} \int_0^{\frac{\lambda_s}{\bar{\lambda}}} \mathrm{d}u \right)] = \mathsf{E}[\int_0^t C_s \lambda_s \mathrm{d}s]
\end{aligned}
$$

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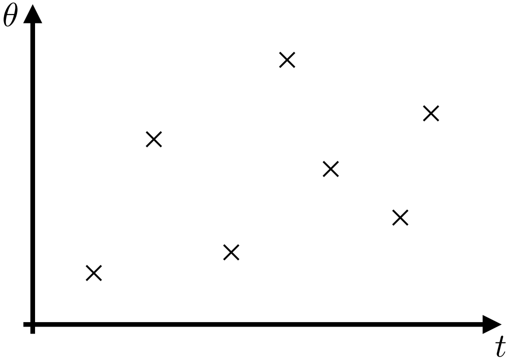
Thinning equation: General case

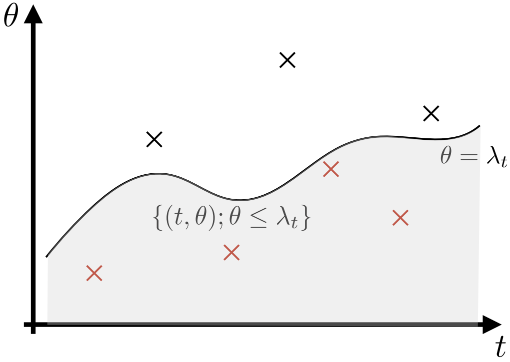


Space-time $(\mathcal{G}_t)$ Poisson measure Q on $\mathbb{R}^+ \times \mathbb{R}^+$ of mean measure $\mathrm{d}t \otimes \mathrm{d}\theta$.

No increasing enumeration of jumps times.

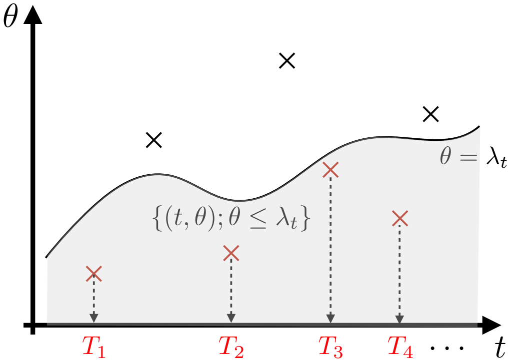
Thinninq equation: General case

Given a predictable process $(\lambda_t)_{t\geqslant 0}$ with $\int_0^t \lambda_s \mathrm{d}s < +\infty$ a.s. $\forall t \geqslant 0$ (nonexplosion condition)



Restriction to predictable subset:

$$\{(s,\theta);\ \theta \leqslant \lambda_s(\omega),\ s \leqslant t\}$$



"Theoretical" Thinning

$N_t^\lambda = \int_0^t \int_{\mathbb{R}+} \mathbb{1}_{\{\theta \leqslant \lambda_s\}} Q(\mathrm{d}s, \mathrm{d}\theta)$ is a *counting process* of $(\mathcal{G}_t)$-intensity λ_t.

Some remarks

- **General case**: Thinning does not give a simulation procedure.
- **Bounded case**: Q can replaced by $\mathbb{1}_{\{u \leqslant \bar{\lambda}\}} Q$ Poisson measure of intensity $\mathrm{d}s \otimes \mathbb{1}_{\{u \leqslant \bar{\lambda}\}} du$.

$\sim$ Compound Poisson process of mean measure $\bar{\lambda} ds \otimes \frac{1}{\bar{\lambda}} \mathbb{1}_{\{u \leqslant \bar{\lambda}\}} du$.

- (λ_s) does not characterizes the distribution of N.

$$N_t^{\lambda} = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{u \leqslant \lambda_s\}} Q(\mathrm{d}s, \mathrm{d}u). \tag{5}$$

- **General case**: Thinning does not give a simulation procedure.
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- Space-time point processes of $(\mathcal{G}_t)$ intensity $\lambda(s, \mathrm{d}x) = \mu(s, x)\gamma(\mathrm{d}x)$:

$$N^{\lambda}(\mathrm{d}t, \mathrm{d}x) = \int_0^t \int_{\mathbb{R}^+ \times E} \mathbb{1}_{\{u \leqslant \mu(s,x)\}} Q(\mathrm{d}s, \mathrm{d}u, \mathrm{d}x).$$

with Q $(\mathcal{G}_t)$ space-time Poisson measure on $\mathbb{R}^+ \times \mathbb{R}^+ \times E$ of mean measure $\mathrm{d}s \otimes \mathrm{d}\theta \otimes \gamma(\mathrm{d}x)$.

SDE driven by Poisson measure

- When $\lambda_t = \alpha(\omega, t, [N]_{t-})$ is a functional of N, (5) $\Rightarrow$ SDE driven by Q:

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \alpha(s,[N^\alpha]_{s-})\}} Q(\mathrm{d}s, \mathrm{d}\theta), \quad dN_t^\alpha = Q(\mathrm{d}t,]0, \alpha(t, [N^\alpha]_{t-})])$$

- **Existence of a well-defined (non-exploding) solution?**

SDE driven by Poisson measure

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- **Existence of a well-defined (non-exploding) solution?**
  - If α is uniformly bounded by a constant $\bar{\lambda}$: same as in bounded case (recursive procedure).
  - Even in simpler case α is not uniformly bounded.
    Example: Linear birth process, $\lambda_t = bN_{t-}$.
- **In the following**: Existence, uniqueness and nonexplosion by strong domination.

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"Converse" result (I)

If a counting process N^{λ} admits a $(\mathcal{G}_t)$-intensity (λ_t),

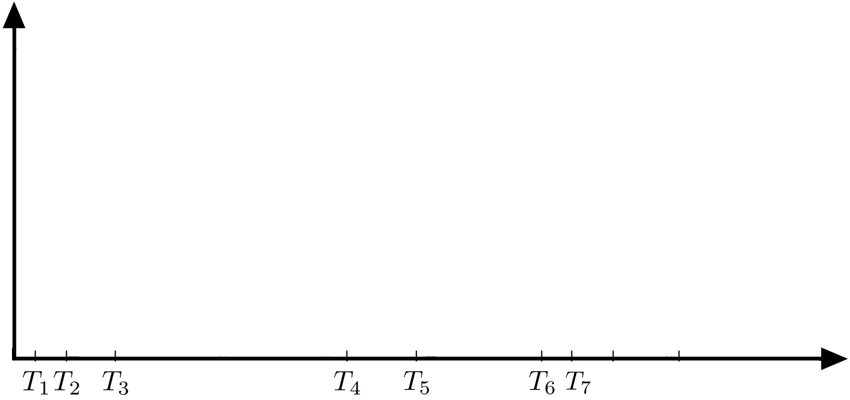
is there a representation of N^{λ} in terms of stochastic integral with respect to some Poisson process?

"Converse" result (I)

If a counting process N^{λ} admits a $(\mathcal{G}_t)$-intensity (λ_t),
is there a representation of N^{λ} in terms of stochastic integral with respect to some Poisson process?

- **YES**: see e.g. Grigelionis (1971), Jacod (1980), Massoulié (1998).
- **Ingredients**:
  - A space-time Poisson measure $\hat{Q}(\mathrm{d}s, \mathrm{d}\theta)$ of intensity $\mathrm{d}s \otimes \mathrm{d}\theta$ and independent of $\mathcal{G}_{\infty}$.
  - A sequence of i.i.d variables $(U_n)_{n \geqslant 0}$ uniform on $[0, 1]$ and independent of $\mathcal{G}_{\infty}$.

"Converse" result (II)



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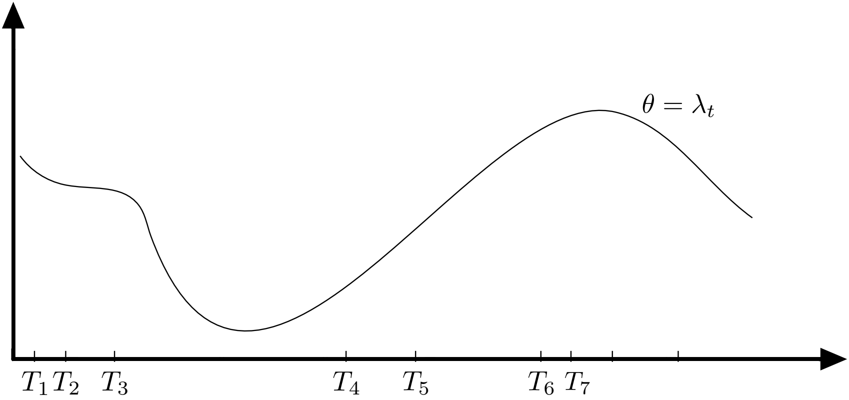

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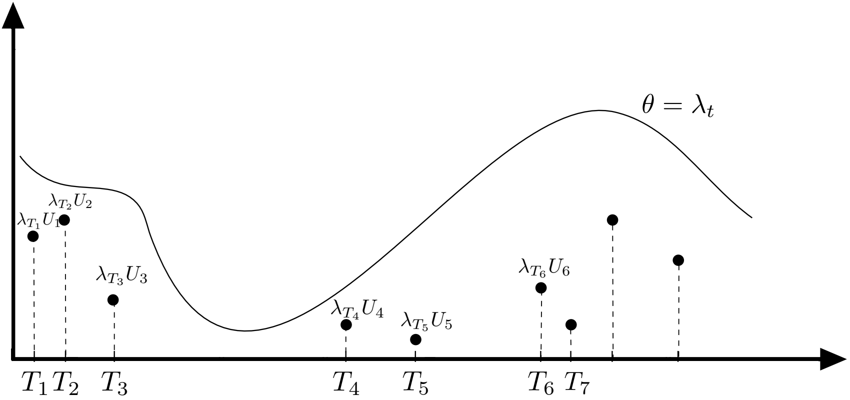

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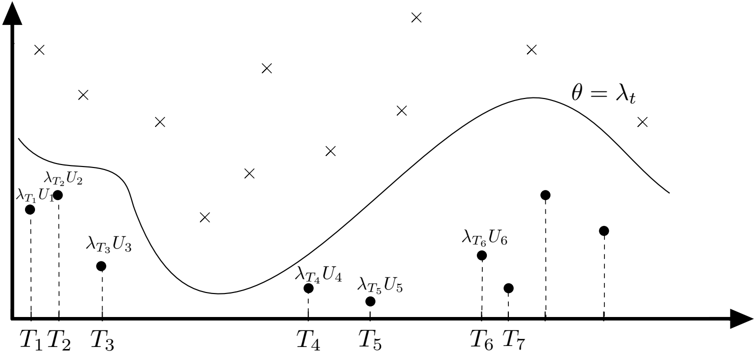

Sketch of the proof

1. Let $Q(\mathrm{d}s, \mathrm{d}\theta) = \sum_{n \geqslant 1} \delta_{T_n}(s)\delta_{(\lambda_{T_n} U_n)}(\theta) + \int_{\mathbb{R}^+ \times \mathbb{R}^+} \mathbb{1}_{\{\theta \geqslant \lambda_s\}} \hat{Q}(\mathrm{d}s, \mathrm{d}\theta)$.
2. Show that $N_t^\lambda = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \lambda_s\}} Q(\mathrm{d}s, \mathrm{d}\theta)$.
3. Show that Q is a space time Poisson measure of mean measure $\mathrm{d}s \otimes \mathrm{d}\theta$.

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A standard nonexplosion criterion

Existence of non-exploding solution to:

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{\theta \leqslant \alpha(s,[N^\alpha]_{s-})\}} Q(\mathrm{d}s, \mathrm{d}\theta), \quad dN_t^\alpha = Q(\mathrm{d}t,]0, \alpha(t, [N^\alpha]_{t-})])?$$

A standard assumption

- Sublinear growth:

$$\alpha(\omega, s, [n]) \leqslant c + bn(s)$$

- Example: Linear Hawkes process ($\alpha(s,[n]) = b + \int_0^s h(s-r)\mathrm{d}n(s)$) with bounded h.
- Under this assumption:

$$\mathrm{E}[N_t^\alpha] = \mathrm{E}[\int_0^t \alpha(s,[N^\alpha]_{s-})\mathrm{d}s] \leqslant \mathrm{E}[\int_0^t (c+bN_{s-}^\alpha)\mathrm{d}s] = ct + \int_0^t b\mathrm{E}[N_{s-}^\alpha]\mathrm{d}s.$$

Gronwall lemma: $\mathrm{E}[N_t^\alpha] \leqslant ce^{bt}$.

Strong comparison of point processes

- Issue: domination by deterministic function, linear assumption can be relaxed (Markov case).
- Existence, uniqueness and nonexplosion by strong domination.

- Comparison of counting processes with ordered intensity processes: Preston (1975), Bhaskaran (1986), Rolski and Szekli (1991), Bezborodov (2015)).
- Pathwise representation: well-adapted to study this problem, simplifies proofs significantly, "strong" pathwise comparison.

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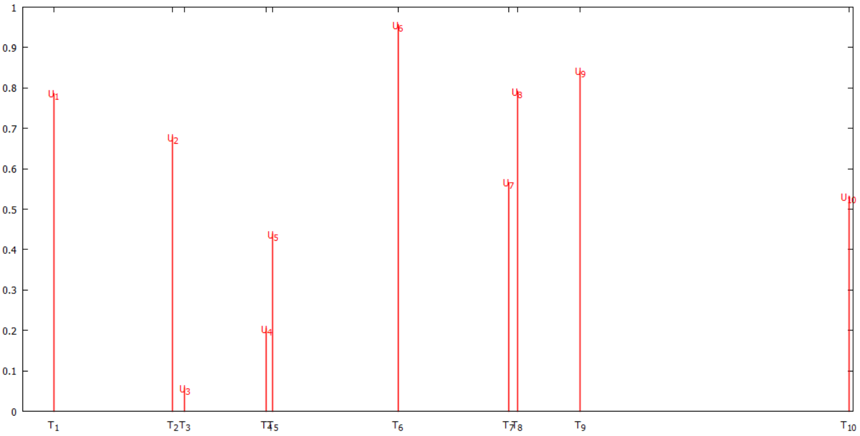
Example NHPs with exponential intensities

$$\lambda^2(t) = e^{-0.2t} \leqslant \lambda^1(t) = e^{-0.1t}.$$



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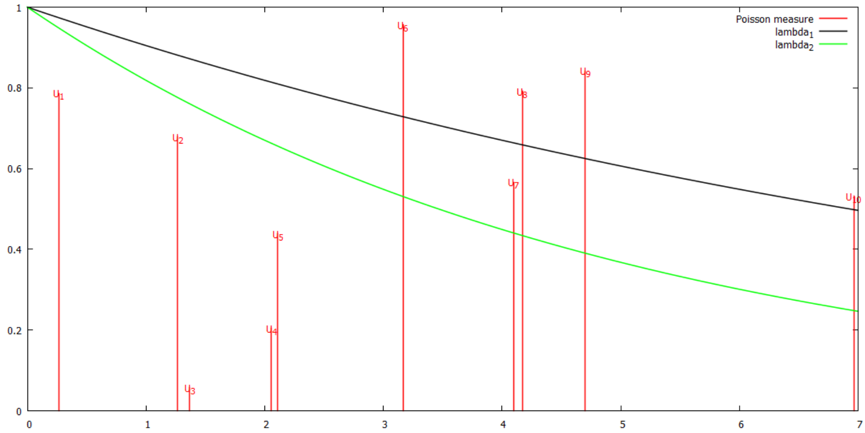

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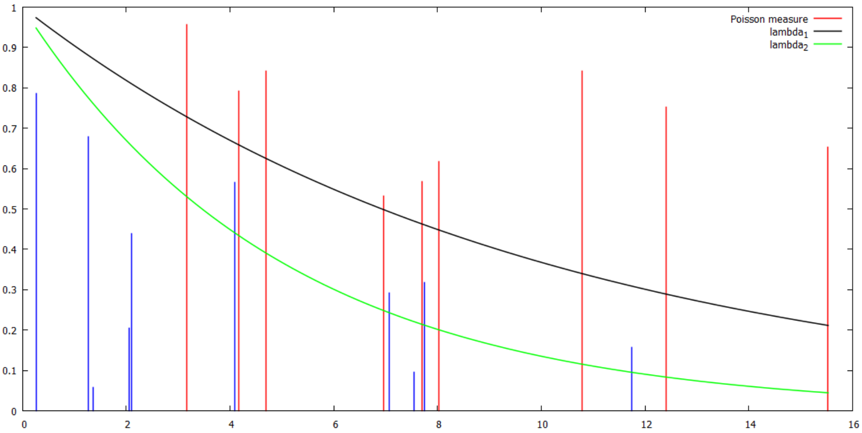

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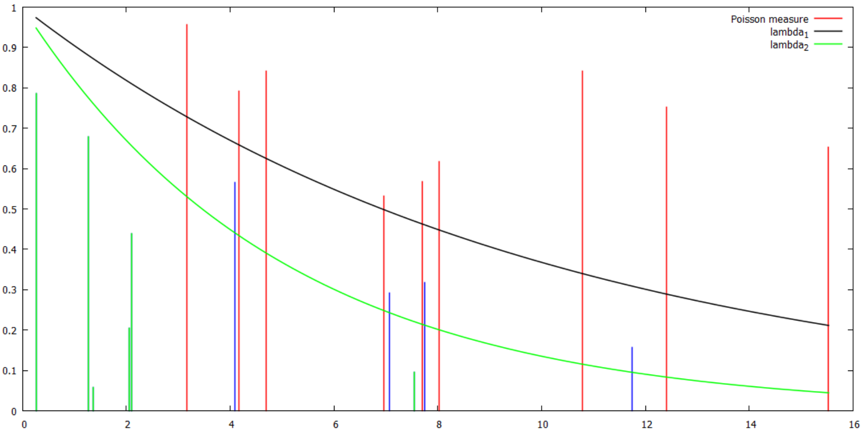

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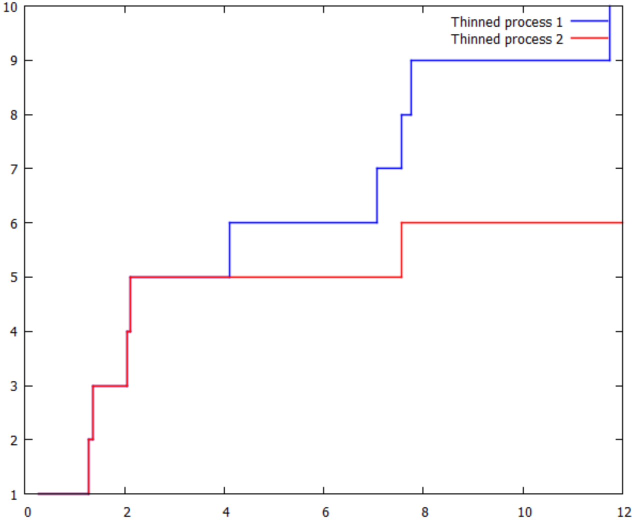

Counting processes with ordered intensity

Let N^1 and N^2 two counting processes with ordered $(\mathcal{G}_t)$-intensities:

$$\lambda_t^2 \leqslant \lambda_t^1 \quad t \geqslant 0,$$

and defined by:

$$N_t^i = \int_0^t \int_{\mathbb{R}^+} \mathbf{1}_{\{u \leqslant \lambda_s^i\}} Q(\mathrm{d}s, d\theta), \quad i = 1, 2.$$

- $\{u \leqslant \lambda_s^2\} \subset \{u \leqslant \lambda_s^1\}$ so that $N_t^2 \leqslant N_t^1, \quad \forall t \geqslant 0$.

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- $\{u \leqslant \lambda_s^2\} \subset \{u \leqslant \lambda_s^1\}$ so that $N_t^2 \leqslant N_t^1, \quad \forall t \geqslant 0$.

Stronger result

- Let $Q^1(\mathrm{d}s, d\theta) = \mathbf{1}_{\{u \leqslant \lambda_s^1\}} Q(ds, d\theta) = \sum_{n \geqslant 1} \delta_{T_n^1}(s) \delta_{\theta_n}(\theta)$,

$$N_t^1 = \int_0^t \int_{\mathbb{R}^+} Q^1(\mathrm{d}s, d\theta), \quad N_t^2 = \int_0^t \int_{\mathbb{R}^+} \mathbf{1}_{\{u \leqslant \lambda_s^2\}} Q^1(\mathrm{d}s, d\theta).$$

$N_t^2 = \sum_{n \geqslant 1} \mathbb{1}_{\{T_n^1 \leqslant t\}} \mathbb{1}_{\{\theta_n \leqslant \lambda_{T_n}^2\}}$ **All jumps of N^2 are jumps of N^1**

Strong domination of space-time point processes

- A space-time point process N^2 is **strongly dominated** by space-time point process N^1, $N^2 \prec N^1$ if
$$N^1 - N^2 \text{ is a space-time point process.}$$
- Example of counting processes:
$$N^2 \prec N^1 \Leftrightarrow \text{all jumps of } N^2 \text{ are jumps of } N^1.$$
- PROPERTY If two counting processes N^1 and N^2 have ordered intensities and are generated with the same Poisson measure, then $N^2 \prec N^1$.

Difficulty:

- When $\lambda_t = \alpha(\omega, t, [N]_{t-})$: natural order of random intensity functionals does not necessary imply an order on intensities processes.

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Strong order of intensity functionals

- Let $\alpha, \beta : \Omega \times \mathbb{R}^+ \times \mathcal{A} \to \mathbb{R}^+$ two $(\mathcal{G}_t)$-predictable functionals.
  α is **strongly majorized** by β, $\alpha \leqslant_s \beta$ if

$$\forall t \geqslant 0, \sup_{[m] \prec [n]} \alpha(t, [m]) \leqslant \beta(t, [n]) \ a.s. \tag{6}$$

- Example $\alpha(t, [n]) = \lambda_t + \int_0^t h_1(t-s)dn(s)$ and
  $\alpha(t, [n]) = \lambda_t + \int_0^t h_2(t-s)dn(s)$ with $0 \leqslant h_1 \leqslant h_2$.

Construction by strong domination (1D case)

Proposition (El Karoui, K.)

Assume that there exists a unique well-defined solution N^β of:

$$\mathrm{d}N_t^\beta = Q(\mathrm{d}t,]0, \beta(t, [N^\beta]_{t-})]).$$

(i) *Then, for all $\alpha \leqslant_s \beta$, there exists a unique solution of the SDE*

$$\mathrm{d}N_t^\alpha = Q\big(\mathrm{d}t,]0, \alpha(t, [N^\alpha]_{t-})]\big). \tag{7}$$

(ii) *Furthermore, N^α is* **strongly dominated** *by N^β ($N^\alpha \prec N^\beta$), i.e. $N^\beta - N^\alpha$ is a counting process (jump times of N^α = jump times of N^β).*

Application to nonexplosion criteria (I)

$$\alpha \leqslant_s \beta \Rightarrow$$

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\theta \leqslant \alpha(t,[N^\alpha]_{t-})} Q(\mathrm{d}s, \mathrm{d}\theta) \prec N_t^\beta = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\theta \leqslant \beta(t,[N^\beta]_{t-})} Q(\mathrm{d}s, \mathrm{d}\theta)$$

- Pathwise comparison result: $N_t^\alpha \leqslant N_t^\beta$, for f nonnegative function

$$\int_0^t f(s) \mathrm{d}N_s^\alpha \leqslant \int_0^t f(s) \mathrm{d}N_s^\beta, \ldots$$

Application to nonexplosion criteria (I)

$$\alpha \leqslant_s \beta \Rightarrow$$

$$N_t^\alpha = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\theta \leqslant \alpha(t,[N^\alpha]_{t-})} Q(\mathrm{d}s, \mathrm{d}\theta) \prec N_t^\beta = \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\theta \leqslant \beta(t,[N^\beta]_{t-})} Q(\mathrm{d}s, \mathrm{d}\theta)$$

- Pathwise comparison result: $N_t^\alpha \leqslant N_t^\beta$, for f nonnegative function

$$\int_0^t f(s)\mathrm{d}N_s^\alpha \leqslant \int_0^t f(s)\mathrm{d}N_s^\beta, \ldots$$

- A FIRST APPLICATION: case $\beta(t,[n]) = c + bn(t)$
  - N^β is a Markov birth process of $(\mathcal{G}_t)$-intensity $(c + bN_t^\beta)$. Its distribution does not depends on Q.
  - $\alpha \leqslant_s \beta$ corresponds to Sublinear growth condition.

$$\Rightarrow N_t^\alpha \leqslant N_t^\beta < \infty, \text{ for all } t \geqslant 0.$$

Application to nonexplosion criteria (II)

Goal Define a class of dominating intensity functionals β associated with nonexploding processes.

- Preceding proof does not rely on the linearity of $\beta(t, [n]) = c + bn(t)$.
- EXTENSION TO NONLINEAR DOMINATING INTENSITIES
  - Markov birth processes ($\beta(t, [n]) = g(n(t))$): n.s.c for nonexplosion

$$\sum_{j=0} \frac{1}{g(j)} = \infty. \tag{8}$$

  - If $\alpha \leqslant_s g$ with g verifying (8), unique nonexploding solution N^α of (7).
  - Sometimes known as the Jacobsen condition (proof without pathwise representation much harder!).

Application to nonexplosion criteria (III)

Goal Define a class of dominating intensity functionals β associated with nonexploding processes.

- A SECOND APPLICATION: if $\alpha \leqslant_s (\lambda_t)$ with $\int_0^t \lambda_s \mathrm{d}s < \infty$, then unique nonexploding solution N^α of (7).

Application to nonexplosion criteria (III)

Goal Define a class of dominating intensity functionals β associated with nonexploding processes.

- A SECOND APPLICATION: if $\alpha \leqslant_s (\lambda_t)$ with $\int_0^t \lambda_s \mathrm{d}s < \infty$, then unique nonexploding solution N^α of (7).
- **Corollary of the proposition**
  If $\alpha \leqslant_s \beta$ with $\beta(t, [n]) = k_t g(n(t))$, (k_t) predictable locally bounded process and g verifying $\sum \frac{1}{g(j)}$, then (7) admits a unique well-defined solution N^α.

Other following results

- **Corollary 2**: $\forall\ \alpha \leqslant_s \beta$, the sequence of jump times of N^β is a localizing sequence of the local martingales $N^\alpha_t - \int_0^t \alpha(s, N^\alpha_s)\mathrm{d}s$.
- **Corollary 3**: Let $(\alpha^i)_{i\in I}$ be a family of intensity functionals with $\alpha^i \leqslant_s \beta$ and:

  $$\mathrm{d}N^i_t = Q(\mathrm{d}t,]0, \alpha^i(t, [N]_{t^-})])$$

  Then $(N^i)_{i\in I}$ is tight.
- Straigthforward extension to strong comparison for space-time point processes.

Sketch of the proof

$\alpha \leqslant_s \beta = N^\beta$,

$$dN_t^\alpha = Q(\mathrm{d}t,]0, \alpha(\omega, t, N_{t-}^\alpha)]) \quad ? \tag{9}$$

- **Step 1**: replace Q by

$$Q^\beta(\mathrm{d}t, \mathrm{d}\theta) = \mathbb{1}_{\{\theta \leqslant \beta(t, [N^\beta]_{t-})\}} Q(\mathrm{d}t, \mathrm{d}\theta) = \sum_{n \geqslant 1} \delta_{T_n^\beta}(\mathrm{d}t)\delta_{\theta_n}(\mathrm{d}s)$$

$$d\tilde{N}_t^\alpha = Q^\beta(\mathrm{d}t,]0, \alpha(\omega, t, \tilde{N}_{t-}^\alpha)]). \tag{10}$$

Jumps times of Q^g can be enumerated increasingly $\Rightarrow$ (10).

- **Step 2**: show equality between (9) and (10).

Remark The proof also gives a simulation algorithm.

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Converse result (I)

Let N^λ with$(\mathcal{G}_t)$- intensity (λ_t) and such that $N^\alpha \prec N^\beta$.

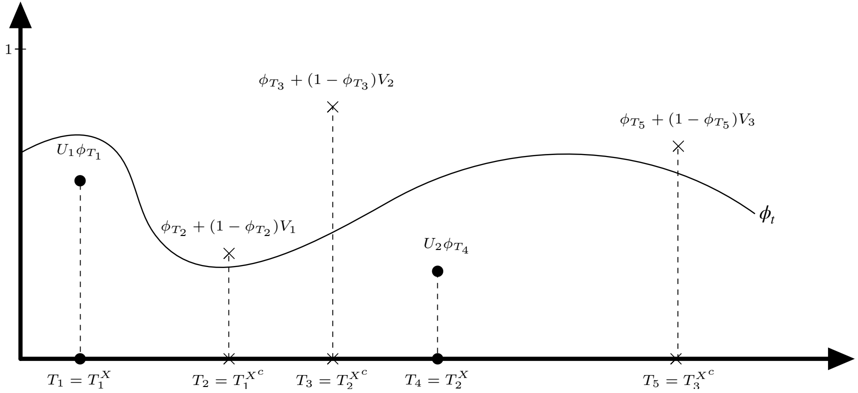
Is there a representation of N^λ in terms of stochastic integral with respect to a marked process with same jumps than N^β?

- **YES**
- **Ingredients**:
  - A sequence of i.i.d variables $(U_n)_{n\geqslant 0}$ uniform on $[0,1]$ and independent of $\mathcal{G}_\infty$.
  - A sequence of i.i.d variables $(V_n)_{n\geqslant 0}$ uniform on $[0,1]$ and independent of $\mathcal{G}_\infty$.
- See also Rolski and Szekli (1991) (distributional viewpoint)

Converse result (II)

Notations:

$$X = N^{\alpha}, \quad X^c = N^{\beta} - N^{\alpha}, \quad \phi_t = \frac{\lambda_t}{\lambda_t^{\beta}} \leqslant 1.$$



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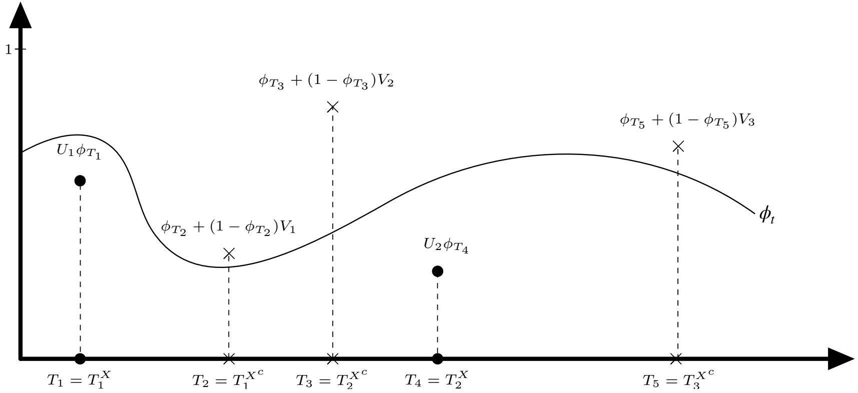


Corollary Two counting processes with the same intensity functional and strongly dominated by the same process have the same distribution.

Outline

1 Some generalities on point processes

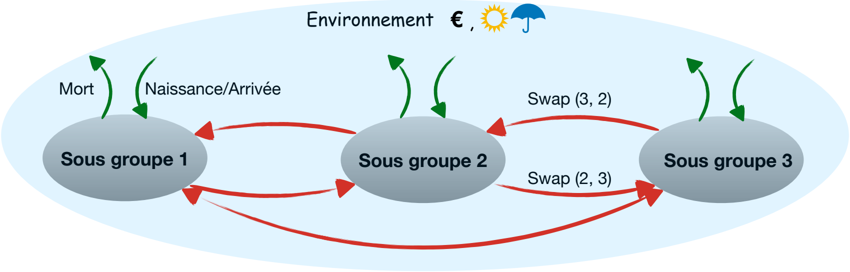
2 Pathwise representation of point processes in random environment

3 Strong comparison of point processes

4 Birth-Death-Swap process in random environment

Birth Death Swap (BDS) systems

An example of BDS system



- The variability of the environment is taken into account $\Rightarrow$ stochastic intensities:

$$\mathrm{P}(\text{ ev of type } \gamma \in]t, t + \mathrm{d}t] | \mathcal{G}_t) \simeq \mu^{\gamma}(\omega, t, Z_t)\mathrm{d}t.$$

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Some additional references (I)

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Some additional references (II)

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